Calculation of Yukawa scattering and impact parameter amplitudes using Legendre Pade approximants

1979 J. Phys. A: Math. Gen. 12 2563

(http://iopscience.iop.org/0305-4470/12/12/033)

J. Phys. A: Math. Gen., Vol. 12, No. 12, 1979. Printed in Great Britain

Calculation of Yukawa scattering and impact parameter amplitudes using Legendre Padé approximants

A K Common

Mathematical Institute, University of Kent, Canterbury, Kent CT2 7NF, UK

Received 12 February 1979

Abstract. The convergence of a certain sequence of Legendre Padé approximants defined in a previous work is illustrated numerically for the case of the Yukawa potential, and in particular when this potential is such that the partial wave series converges only slowly. The sequences converge appreciably faster than this series in the physical region, and also converge quite quickly even in regions where the series diverges. The Legendre Padé approximants are used to construct convergent sequences of approximants to the impact parameter amplitude, and numerical results are given in the Yukawa case. The unitarity of the extrapolated partial waves is also studied numerically.

1. Introduction

In previous works (Common and Stacey 1978a, b, 1979) convergent sequences of Legendre Padé approximants were constructed to both the real and imaginary parts of scattering amplitudes for a large class of potentials, including the important Yukawa potential $V(r) = -G \exp(-\mu r)/r$. It is of course interesting to see how well these sequences of approximants converge in practice, and in this work we present results of numerical evaluations for the Yukawa case.

In § 2 we present sets of results for both the real and imaginary parts of the scattering amplitude at various points in the cut plane analyticity domain of the amplitude and compare them with the truncated partial wave series.

It is well known (Henzi 1966, Kupsch and Statamescu 1973, Islam 1976) that, in order to define precisely the impact parameter amplitude, one needs to know the scattering amplitude not only for physical values of $z = \cos\theta$, where θ is the scattering angle, but also for values extending to ∞ in some direction in the complex plane. Therefore, to evaluate the impact parameter amplitude from physical values of the scattering amplitude, we need a method for continuing this amplitude into the complex plane. Our approximants provide such a technique, and, when used in the definition of the impact amplitude given by Henzi (1966), give approximants to this amplitude. It is shown in § 3 that these approximants have a simple analytic form and converge to the exact impact parameter amplitude when the corresponding Legendre Padé approximants converge to the scattering amplitudes. Numerical results are presented for the case of the Yukawa amplitude.

A useful property of our Legendre Padé approximants is that the higher partial waves have a simple expression in terms of the parameters of the approximant. An interesting question is how well these extrapolated partial waves satisfy unitarity. We have investigated this problem numerically for the Yukawa amplitude, and the results are presented in § 4. It is found that, although these extrapolated partial waves do not satisfy unitarity precisely, it is approximately satisfied when the approximant is close to the scattering amplitude in the physical region and when the partial waves are not too small.

Finally, we summarise the conclusions of this work in § 5.

2. Legendre Padé approximants to the Yukawa scattering amplitude

We consider scattering† by the Yukawa potential $V(r) = -G \exp(-\mu r)r$. The phase shifts were evaluated using the subroutine PHASE constructed by Klozenberg (1974).

In table 1 we present the values of the first 15 phase shifts for $G = 4$, $\mu = 1$ and the energy of the scattered particle $\equiv s = k^2 = 45$. It will be seen from table 1 that for these values of the parameters the partial waves are only slowly tending to zero, so that the partial wave series will converge only slowly in the physical region. The full domain of convergence is a very small ellipse with foci at ± 1 and right-hand extremity at $z = 1 + (\mu^2/2k^2)(4\mu^2 + \mu^4/k^2) \simeq 1{\cdot}045$, even if the first Born approximation is subtracted off, so the series cannot be used directly to give a good continuation away from the physical region.

Table 1. The partial wave phase shifts.

l	0	1	2	3	4	5	6
δ_l(rad)	0·7912	0·4978	0·3567	0·2693	0·2092	0·1655	0·1326
l	7	8	9	10	11	12	13
δ_l(rad)	0·1073	0·0874	0·0161	0·0589	0·0487	0·0404	0·0336

The Legendre Pade approximants which form convergent sequences of approximants to the real and imaginary parts of the Yukawa scattering amplitude are defined as follows. First of all, for the imaginary part of the scattering amplitude,

$$\operatorname{Im} f(s,t) = \sum_{l=0}^{\infty} (2l+1) \operatorname{Im} f_l(s) P_l(z), \qquad z = 1 + \frac{t}{2k^2}, \tag{2.1}$$

the approximants are‡

$$\operatorname{Im} f_n^{\mathrm{L}}(z) = \sum_{p=1}^{2n} \frac{\alpha'_p(1-\sigma_p^2)}{(1-2\sigma_p z + \sigma_p^2)^{3/2}} + \sum_{l=0}^{L_0-1} (2l+1)\Big(\operatorname{Im} f_l(s) - \sum_{p=1}^{2n} \alpha'_p \sigma_p^l\Big) P_l(z), \tag{2.2}$$

where $\alpha'_p = \alpha_p \sigma_p^{-L_0-1}$ and

$$g_n(w) = \sum_{p=1}^{2n} \frac{\alpha_p}{1+\sigma_p w} \tag{2.3}$$

† We choose units where $\hbar = c = 1 = 2 \times$ the mass of the scattered particle.
‡ The dependence on s of the terms is understood.

is the approximant to

$$g(w) \equiv \sum_{l=0}^{\infty} \operatorname{Im} f_{l+L_0}(-w)^l \tag{2.4}$$

defined in equation (5.4) of a previous paper (Common and Stacey 1978b).

Similarly, for the real part of the scattering amplitude,

$$\operatorname{Re} f(s,t) = \sum_{l=0}^{\infty} (2l+1) \operatorname{Re} f_l(s) P_l(z), \qquad z = 1 + \frac{t}{2k^2}, \tag{2.5}$$

the approximants are†

$$\operatorname{Re} f_n^{\mathrm{L}}(z) = \sum_{p=1}^{2n} \frac{\alpha'_p(1-\sigma_p^2)}{(1-2\sigma_p z+\sigma_p^2)^{3/2}} + \sum_{l=0}^{L_0-1} (2l+1)\left(\operatorname{Re} f_l(s) - \sum_{p=1}^{2n} \alpha'_p \sigma_p^l\right) P_l(z), \tag{2.6}$$

where now

$$g_n(w) = \sum_{p=1}^{2n} \frac{\alpha_p}{1+\sigma_p w} \tag{2.7}$$

is the approximant to

$$g(w) = \sum_{l=0}^{\infty} \operatorname{Re} f_{l+L_0}(s)(-w)^l \tag{2.8}$$

analogous to the approximant (2.3). Previously (Common and Stacey 1979) the first Born term was subtracted from $\operatorname{Re} F(s,t)$ in order to prove convergence of the corresponding approximants. However, we found that our numerical results were more stable if this subtraction was not made.

Finally, L_0 appearing in (2.2), (2.6) is the number of subtractions in the fixed s-dispersion relations for the respective amplitudes. The values to be taken for L_0 are related to the positions of the Regge poles of the Yukawa amplitude as described previously (Common and Stacey 1979). A study of the trajectories of these poles in the case of the Yukawa potential has been made by Lovelace and Masson (1962), and from their figure 1 we can deduce that for $s=45$, $G=4$ we can take $L_0=1$ in both (2.2) and (2.6).

In table 2 we give the values of $\operatorname{Re} f_n^{\mathrm{L}}(z)$ and $\operatorname{Im} f_n^{\mathrm{L}}(z)$ for real values of z and for $n=3$ and 4. We compare them with those of the partial wave series $f_{\mathrm{TS}}(z)$ truncated at $l=15$. In doing this we have omitted values of $f_{\mathrm{TS}}(z)$ outside the physical region, since the partial series blows up outside its ellipse of convergence, which has foci at ± 1 and right-hand extremity at $z=1{\cdot}045$ as mentioned earlier.

It will be seen that our approximants converge appreciably faster than the partial wave series in the physical region $-1 \leqslant z \leqslant 1$, and also a long way along the negative z axis where the series blows up. To illustrate how our approximants converge in the complex plane, we give in table 3 values of our Legendre Padé approximants $F_n^{\mathrm{L}}(z)$ to the whole scattering amplitude $f(s,t)$ rather than to the real and imaginary parts separately. Although we cannot *prove* convergence for this type of approximant, in practice we find that they seem to converge as well as the approximants $\operatorname{Re} f_n^{\mathrm{L}}(z)$ and $\operatorname{Im} f_n^{\mathrm{L}}(z)$. This of course happens in the case of ordinary Padé approximants, which in practice converge for many series for which convergence cannot be proved. The $F_n^{\mathrm{L}}(z)$ are computed at points on an ellipse with foci at ± 1 and left-hand extremity $z=-2{\cdot}51$.

† The dependence on s of the terms is understood.

Table 2. Approximants to the real and imaginary part of the scattering amplitude on the real axis compared with the truncated partial wave series.

z	$\mathrm{Re}\, f_3^{\mathrm{L}}(z)\times 10^2$ $\mathrm{Im}\, f_3^{\mathrm{L}}(z)\times 10^2$	$\mathrm{Re}\, f_4^{\mathrm{L}}(z)\times 10^2$ $\mathrm{Im}\, f_4^{\mathrm{L}}(z)\times 10^2$	$\mathrm{Re}\, f_{\mathrm{TS}}(z)\times 10^2$ $\mathrm{Im}\, f_{\mathrm{TS}}(z)\times 10^2$
−5·32	−0·2615	−0·2809	
	0·7076	0·7008	
−2·51	−0·2338	−0·2393	
	1·3049	1·3033	
−1·32	−0·9810	−0·9900	
	1·9868	1·9866	
−1·00	−0·0050	−0·0045	16·457
	2·3078	2·3078	2·4525
−0·54	0·2421	0·2422	−0·0359
	2·9848	2·9848	2·9761
−0·06	0·8408	0·8407	−0·7615
	4·2169	4·2169	4·1766
0·54	4·2344	4·2345	6·863
	8·6343	8·6342	8·6979
0·93	45·6360	45·7469	36·383
	29·3516	29·3480	29·1808
1·00	225·896	224·542	230·912
	51·8826	51·8817	51·8099

Table 3. The real and imaginary parts of approximants to the scattering at points on an ellipse with foci at ±1 and semi-major axis 2·51.

x y	$\mathrm{Re}\, F_3^{\mathrm{L}}(z)\times 10^2$ $\mathrm{Im}\, F_3^{\mathrm{L}}(z)\times 10^2$	$\mathrm{Re}\, F_4^{\mathrm{L}}(z)\times 10^2$ $\mathrm{Im}\, F_4^{\mathrm{L}}(z)\times 10^2$	$\Delta(z)(\%)$
−2·51	−0·2320	−0·2373	0·464
0·00	1·3085	1·3077	
−2·43	−0·4608	−0·4665	0·466
0·57	1·3294	1·3317	
−2·20	−0·7165	−0·7241	0·492
1·11	1·3353	1·3413	
−1·83	−1·0160	−1·0177	0·553
1·58	1·3191	1·3295	
−1·34	−1·3815	−1·3762	0·677
1·94	1·2658	1·2809	
−0·78	−1·8420	−1·8218	0·924
2·19	1·1481	1·1670	
−0·16	−2·4363	−2·3837	1·433
2·30	0·9231	0·9402	
0·47	−3·2235	−3·0926	2·574
2·26	0·5371	0·5281	
1·07	−4·3348	−3·9833	5·466
2·08	−0·0053	−0·1501	

This ellipse is of course much larger than the ellipse of convergence of the partial wave series. The approximants are given for $n=3$ and 4, and we have computed the fractional error

$$\Delta(z)=|F_4^{\mathrm{L}}(z)-F_3^{\mathrm{L}}(z)|/|F_3^{\mathrm{L}}(z)|$$

which is given in table 3 as a percentage. It will be seen that the convergence is quite good well away from the cut of $F(z) \equiv f(s, t)$ which runs from $z = 1{\cdot}045$ to ∞, but deteriorates as this cut is approached. As expected, we find that as we go to larger ellipses the convergence is slower.

3. Approximants to the impact parameter amplitude

The impact parameter amplitude $a(b, s)$ has the standard definition

$$a(b, s) = \frac{1}{2k^2} \int_{-\infty}^{0} dt\, f(s, t) J_0(b\sqrt{-t}), \qquad b \geqslant 0, \tag{3.1}$$

but difficulties occur when $f(s, t)$ does not decrease faster than $(-t)^{-1/2}$ on the negative t axis, since the integral in (3.1) is then no longer convergent. In such cases the impact parameter amplitude may be split into a distribution at $b = 0$ plus a well-behaved function for $b > 0$ (Henzi 1966, Kupsch and Statamescu 1973, Islam 1976).

To define $a(b, s)$ uniquely, all the values of $f(s, t)$ on a contour extending to infinity in the complex t plane must be used (or equivalent information). Often $f(s, t)$ is only known for the physical values $-4k^2 \leqslant t \leqslant 0$, and one has therefore to continue it outside this region to evaluate $h(b, s)$. An obvious method of continuation is to use our Legendre Padé approximants, and it turns out that the corresponding approximants to $h(b, s)$ have a simple analytic form.

For convenience we concentrate our discussion on $a(b, s)$, the impact parameter amplitude corresponding to $\operatorname{Im} f(s, t)$ as defined by Henzi (1966),

$$a(b, s) = \frac{1}{4\pi k^2 \mathrm{i}} \int_C dt\, \operatorname{Im} f(s, t) K_0(b\sqrt{t}), \tag{3.2}$$

where C is the contour indicated in figure 1 and $K_0(z)$ is the modified Bessel function. In that case

$$\operatorname{Im} f(s, t) = \sum_{n=0}^{L_0 - 1} a_n(s) t^n + 2k^2 \int_0^\infty a(b, s) J_0^{L_0}(b\sqrt{-t}), \tag{3.3}$$

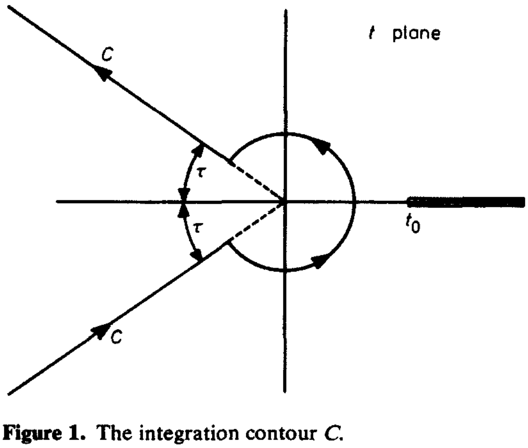

Figure 1. The integration contour C.

with

$$J_0^{L_0}(z)=\sum_{m=L_0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}. \tag{3.4}$$

We define the following sequence of approximants to $a(b, s)$,

$$a_n(b, s)=\frac{1}{4\pi k^2\mathrm{i}}\int_C \mathrm{d}t\ \mathrm{Im}\, f_n^{\mathrm{L}}(z)\mathrm{K}_0(b\sqrt{t}), \qquad n=0, 1, 2, \ldots, \quad b>0, \tag{3.5}$$

where $\mathrm{Im}\, f_n^{\mathrm{L}}(z)$ has been defined in (2.2) and $z=1+t/2k^2$.

Theorem 3.1. If $\mathrm{Im}\, f_n^{\mathrm{L}}(z)$ converges uniformly to $\mathrm{Im}\, f(s, t)$ on any bounded interval of C, then

$$\lim_{n\to\infty} a_n(b, s)=a(b, s), \qquad b>0,$$

and the limit is uniform for s fixed and all $b\geq b_0>0$, where b_0 is a positive constant.

Proof. The result follows from the assumed convergence of $\mathrm{Im}\, f_n^{\mathrm{L}}(z)$ to $\mathrm{Im}\, f(s, t)$ and the inequality

$$|\mathrm{K}_0(b\sqrt{t})|<c_1\exp(-b\sqrt{|t|}\sin\tfrac{1}{2}\tau), \qquad b>0, \quad t\in C, \tag{3.6}$$

where c_1 is a constant.

We will now obtain an explicit form for $a_n(b, s)$ when

$$|\sigma_p|<1, \qquad -\pi<\arg\sigma_p<\pi, \quad p=1, \ldots, n \tag{3.7a}$$

or

$$|\sigma_p|\geq 1, \qquad -\tfrac{1}{2}\pi\leq\arg\sigma_p\leq\tfrac{1}{2}\pi, \quad p=1, \ldots, n. \tag{3.7b}$$

Theorem 3.2. If the approximant $\mathrm{Im}\, f_n^{\mathrm{L}}(z)$ to $\mathrm{Im}\, f(s, t)$ is such that the conditions (3.7) are satisfied, then

$$a_n(b, s)=\frac{1}{2}\sum_{p=1}^{2n}\frac{\alpha'_p(1-\sigma_p^2)}{\sigma_p^{3/2}\chi_p}\mathrm{e}^{-\chi_p bk}, \qquad b>0, \tag{3.8}$$

where

$$\chi_p^2=(1-\sigma_p^2)/\sigma_p, \qquad \text{with} \quad \mathrm{Re}\,\chi_p\geq 0.$$

Proof. It is straightforward to show that the terms in $\mathrm{Im}\, f_n^{\mathrm{L}}(z)$ proportional to $P_l(z)$ do not contribute to the contour integral on the RHS of (3.5). This follows by closing C by an arc of a large circle, such that on the arc $|\arg t|<\tau$. The integral over this arc tends to 0 as the radius of the arc tends to ∞ because of (3.6), and therefore so does the integral of these terms over C, since they are analytic inside the closed contour. Therefore

$$a_n(b, s)=\frac{1}{4\pi\mathrm{i}k^2}\sum_{p=1}^{2n}\alpha'_p(1-\sigma_p^2)\int_C\frac{\mathrm{d}t\ \mathrm{K}_0(b\sqrt{t})}{(1-2\sigma_p z+\sigma_p^2)^{3/2}}. \tag{3.9}$$

Since we assume in (3.7) that σ_p does not lie on the negative real axis, we can choose τ so that none of the cuts of the approximants lie in the sectors $-\pi<\arg t\leq-\pi+\tau$,

$\pi - \tau < \arg t \leq \pi$. In that case the contour can be collapsed onto the negative t axis, and using the well-known discontinuity of $K_0(b\sqrt{t})$ across this axis (Abramowitz and Stegun 1964, p 375),

$$a_n(b, s) = -\frac{1}{4k^2} \sum_{p=1}^{2n} \alpha'_p(1-\sigma_p^2) \int_0^{-\infty} \frac{J_0(b\sqrt{|t|})\,dt}{(1-2\sigma_p z+\sigma_p^2)^{3/2}}. \tag{3.10}$$

The square root in the denominator of the integrand has by convention a positive real part, and we show in the Appendix that, if σ_p satisfies (3.7) for all $p = 1, \ldots, n$, then we may write

$$a_n(b, s) = +\frac{1}{2} \sum_{p=1}^{2n} \frac{\alpha'_p(1-\sigma_p^2)}{(bk)^{1/2}\sigma_p^{1/2}} \int_0^{\infty} \frac{u^{1/2}(buk)^{1/2}J_0(buk)\,du}{[(1-\sigma_p)^2/\sigma_p+u^2]^{3/2}}, \tag{3.11}$$

with $-\pi < \arg \sigma < \pi$, $-\pi < \arg[(1-\sigma_p)^2/\sigma_p + u^2] < \pi$. The integrand is a standard integral of a Bessel function, and substituting in the form for it given by equation (8) of Bateman (1954, p 7) we arrive at the expression (3.8) for $a_n(b, s)$.

We have had to require that the σ_p satisfy (3.7) in order that the argument of $(1-\sigma_p)^2/\sigma_p + u^2$ lies between $-\pi$, π so that the standard form of the integral may be used. These conditions will certainly be satisfied if $g(w) = \Sigma_{l=0}^{\infty} \operatorname{Im} f_{l+L_0+1}(s)(-w)^l$ has a radius of convergence $r_0 > 1$, since the poles of $g_n(w)$ will migrate to the cut of $g(w)$ as $n \to \infty$. This will be the case for scattering by classes of potentials considered previously (Common and Stacey 1978a, b, 1979), including the important pure Yukawa potential.

We give in table 4 values of $a_n(b, s)$ with $n = 3, 4, 5$ for scattering by the Yukawa potential considered in § 2 for a selection of values of b, and it can be seen that the approximants are converging quickly. It is well known that for large bk, $a(b, s) \simeq \operatorname{Im} f_l(s)$ with $l = bk - \frac{1}{2}$. This is why we have chosen the particular values of b in table 4, which correspond to $l = 1, 2, \ldots$ as illustrated. For comparison we have also given $\operatorname{Im} f_l(s)$ at these l values. Comparing $a_n(b, s)$ with $\operatorname{Im} f_l(s)$, it can be seen in this example that the impact parameter amplitude approximates the partial wave amplitude for even quite low values of l.

Table 4. The approximate impact parameter amplitudes as compared with those of Cottingham and Peierls and the partial wave amplitude.

l	b	$a_3(b,s)\times 10^2$	$a_4(b,s)\times 10^2$	$a_5(b,s)\times 10^2$	$a_{CP}(b,s)\times 10^2$	$\operatorname{Im} f_l(s)\times 10^2$
1	0·224	23·4290	23·4066	23·3952	38·43	22·8000
2	0·373	12·3192	12·3217	12·3232	17·46	12·1922
3	0·522	7·1120	7·1123	7·1122	7·745	7·0783
4	0·671	4·3185	4·3181	4·3180	6·228	4·3135
5	0·820	2·7103	2·7104	2·7105	3·682	2·7149
6	0·960	1·7421	1·7423	1·7423	1·585	1·7488
7	1·118	1·1396	1·1401	1·1401	1·645	1·1464
8	1·267	0·7550	0·7565	0·7565	1·141	0·7618
9	1·416	0·5048	0·5076	0·5075	0·301	0·5119
11	1·714	0·2295	0·2349	0·2344		0·2371
13	2·013	0·1057	0·1119	0·1111		0·1127
15	2·311	0·0490	0·0546	0·0536		0·0546

An earlier way of dealing with the fact that one needed the scattered amplitude for values of t extending to infinity to obtain the impact parameter amplitude was to take instead an integral over only physical values of t (Cottingham and Peierls 1965) and define

$$a_{\mathrm{CP}}(b, s)=\frac{1}{2k^2}\int_{-4k^2}^{0} \mathrm{J}_0(b\sqrt{t})\,\mathrm{Im}\, f(s, t)\,\mathrm{d}t. \tag{3.12}$$

We have evaluated these amplitudes by doing the integration numerically. $\mathrm{Im}\, f(s, t)$ was calculated by summing the truncated partial wave series up to $l=14$, which gave $a_{\mathrm{CP}}(b, s)$ to within an accuracy of 1% for $b\leqslant 1{\cdot}4$. For higher values of b the results become unstable and have been omitted from table 4. It will be seen that there is an appreciable difference between the values of the impact parameter amplitude defined in the above manner and that defined in (3.2).

4. Extrapolated partial waves

As is the case for the ordinary Padé approximants, our approximants to the scattering amplitude appear to converge for a much wider class of amplitudes than that for which convergence can be proved. For instance, as mentioned in § 2, we can construct approximants to the full amplitude $f(s, t)$. They are

$$F_n^{\mathrm{L}}(z)\equiv\sum_{p=1}^{n}\frac{\alpha'_p(1-\sigma_p^2)}{(1-2\sigma_p z+\sigma_p^2)^{3/2}}+\sum_{l=0}^{L_0-1}(2l+1)\Big(f_l(s)-\sum_{p=1}^{n}\alpha'_p\sigma_p^l\Big)P_l(z),$$
$$n=0, 1, 2, \ldots, \tag{4.1}$$

where $\alpha'_p=\alpha_p\sigma_p^{l-L_0-1}$ are related in the usual way to the $[n-1/n]$ Padé approximant to $g(w)=\Sigma_{l=0}^{\infty} f_{l+L_0}(-w)^l$. Since here we are not concerned with proving convergence, we do not add and subtract a function to $g(w)$, and so the sum over p runs only to n and not $2n$ as in (2.2) and (2.6).

A useful property of our approximants is that their partial waves are easily evaluated. For example, if $f_{l,n}^{\mathrm{A}}(s)$ is the lth partial wave of $F_n^{\mathrm{L}}(z)$, then using the fact that the terms in the sum over p are just proportional to the generating functions of the Legendre polynomials,

$$f_{l,n}^{\mathrm{A}}(s)=\sum_{p=1}^{n}\alpha'_p(1-\sigma_p^2)\sigma_p^l, \qquad l\geqslant L_0. \tag{4.2}$$

Since

$$f_{l,n}^{\mathrm{A}}(s)-f_l(s)=\frac{1}{2}\int_{-1}^{1}\{F[s, 2s(z-1)]-F_n^{\mathrm{L}}(z)\}P_l(z)\,\mathrm{d}z, \tag{4.3}$$

it follows that $f_{l,n}^{\mathrm{A}}(s)$ will be a good approximant to $f_l(s)$ if the average error in $F_n^{\mathrm{L}}(z)$ is much less than $f_l(s)$ itself. We have shown previously (Common and Stacey 1978a) that the $f_{l,n}^{\mathrm{A}}(s)$ are exact for $L_0\leqslant l\leqslant L_0+2n$, and we expect them to be a good approximation for higher values of l if the $f_l(s)$ do not decrease too quickly with increasing l.

In table 5 we give values of the real and imaginary parts of the partial waves of the approximants $F_4^{\mathrm{L}}(z)$ and compare them with exact partial waves $f_l(s)$. The partial

Table 5. Extrapolated partial waves and their unitarity as compared with the exact partial waves.

l	$\mathrm{Re}\, f_l \times 10$	$\mathrm{Im}\, f_l \times 10^2$	$\mathrm{Re}\, f_{l,4}^{\mathrm{A}} \times 10$	$\mathrm{Im}\, f_{l,4}^{\mathrm{A}} \times 10^2$	$\mathrm{Im}\, f_{l,4}^{\mathrm{A}}/\lvert f_{l,4}^{\mathrm{A}}\rvert^2$
9	0·7136	0·5119	0·7137	0·5117	0·9995
10	0·5881	0·3471	0·5882	0·3461	0·9966
11	0·4864	0·2371	0·4867	0·2344	0·9872
12	0·4035	0·1630	0·4040	0·1575	0·9636
13	0·3356	0·1127	0·3363	0·1035	0·9144

waves for $l \leqslant 8$ agree exactly, and we have presented the values for $9 \leqslant l \leqslant 13$. It will be seen from table 5 that for these values of l the extrapolated partial waves approximate well the exact waves, and from the last column we also see that they satisfy the unitarity condition quite well. For higher values of l the partial waves of the exact amplitude continue to decrease, and the $f_{l,4}^{\mathrm{A}}$ become relatively more inaccurate as the size of the partial wave becomes of the same order as the error between $F_4^{\mathrm{L}}(z)$ and $f(s, t)$, as we would expect.

5. Conclusions

We have demonstrated in this work, using the important example of Yukawa scattering, the usefulness of Legendre Padé approximants in evaluating the scattering amplitude from partial waves when these waves decrease only slowly with l. Not only do the Legendre Padé approximants converge much faster than the partial wave series in the physical region, but also give reasonable convergence well outside the ellipse of convergence of the series.

As a consequence, we obtain perhaps the most important result of this work, a convergent sequence of approximants to the true impact parameter amplitude, which has the extremely simple explicit form given in equation (3.8). For the Yukawa scattering we have given results in table 4 which demonstrate that our approximants to the impact parameter amplitude converge quickly for values of the impact parameter up to twice the range of the potential.

We have shown in § 4 that the higher partial waves for our approximants, which have the simple form (4.2), give good approximants to the higher partial waves of the exact amplitude. It has been our experience with the subroutine PHASE that it tends to break down if one tries to calculate too many partial waves. Thus our approximants could have an important role in evaluating such partial waves.

Acknowledgments

I am most grateful to Dr J P Whelan (formerly Dr Klozenberg) for sending me a copy of his subroutine PHASE, and to Mr T Stacey for use of his subroutines for calculating Legendre Padé approximants.

I would also like to thank Miss Diane Mayes for her efficient typing of this paper and previous papers in the series, and to Mr John Pell for his excellent drawing of all the figures in these papers.

Appendix

We wish to show that if

$$-\pi < \arg[u^2 + (1-\sigma_p)^2/\sigma_p] < \pi \tag{A1}$$

and

$$|\sigma_p| \leqslant 1, \qquad -\pi < \arg \sigma_p < \pi \tag{A2}$$

or

$$|\sigma_p| > 1, \qquad -\tfrac{1}{2}\pi \leqslant \arg \sigma_p \leqslant \tfrac{1}{2}\pi, \tag{A3}$$

then

$$-\pi < \arg[u^2\sigma_p + (1-\sigma_p)^2] < \pi, \tag{A4}$$

i.e.

$$-\pi < \arg[1 - 2\sigma_p(1 + t/2k^2) + \sigma_p^2] < \pi, \tag{A5}$$

where

$$t = -k^2u^2.$$

Suppose $|\sigma_p| = \rho \leqslant 1$, $0 \leqslant \arg \sigma_p = \phi < \pi$. Then

$$\arg[u^2\sigma_p + (1-\sigma_p)^2] = \arg \sigma_p + \arg[u^2 + (1-\sigma_p)^2/\sigma_p] \geqslant -\pi \tag{A6}$$

from (A1), so the left-hand inequality of (A4) is satisfied.

Also

$$\arg[u^2 + (1-\sigma_p)^2/\sigma_p] = \arg[u^2 - 2 + (\rho + 1/\rho)\cos\phi + \mathrm{i}(\rho - 1/\rho)\sin\phi]. \tag{A7}$$

Then, since we are assuming $0 < \rho \leqslant 1$, $0 \leqslant \phi < \pi$,

$$\arg[u^2 + (1-\sigma_p)^2/\sigma_p] \leqslant 0 \leqslant \pi - \phi. \tag{A8}$$

Therefore

$$\arg[u^2\sigma_p + (1-\sigma_p)^2] < \phi + \pi - \phi = \pi, \tag{A9}$$

so the right-hand side of (A4) is satisfied.

The proof of (A4) when $-\pi < \phi < 0$, $\rho \leqslant 1$ or when $-\frac{1}{2}\pi \leqslant \phi \leqslant \frac{1}{2}\pi$, $\rho > 1$ follows in exactly the same way.

References

Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* (New York: Dover)
Bateman H 1954 *Tables of Integral Transforms* vol II (New York: McGraw-Hill)
Common A K and Stacey T 1978a *J. Phys. A: Math. Gen.* **11** 259–73
—— 1978b *J. Phys. A: Math. Gen.* **11** 275–89
—— 1979 *J. Phys. A: Math. Gen.* **12** 1399–417
Cottingham W N and Peierls R 1965 *Phys. Rev.* B **137** 147–54
Henzi R 1966 *Nuovo Cim.* A **46** 370–406
Islam M M 1976 *Nucl. Phys.* B **104** 511–32
Klozenberg J P 1974 *J. Phys. A: Math. Gen.* **9** 1840–6
Kupsch J and Statamescu I O 1973 *Nuovo Cim.* A **15** 663–88
Lovelace C and Masson D 1962 *Nuovo Cim.* **26** 472–84